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# SUBSPACE THEOREMS FOR HOMOGENEOUS POLYNOMIAL FORMS

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ABSTRACT

We prove a subspace theorem for homogeneous polynomial forms which generalizes Schmidt's subspace theorem for linear forms. Further, we formalize the subspace theorem into a form which is just the counterpart of a second main theorem in Nevanlinna's theorem, and also suggest a problem.

## 1. Introduction

Let $\kappa$ be a number field and let $\bar{\kappa}$ be an algebraic closure. Let $M_\kappa$ be the canonical set of distinct inequivalent valuations (or places) of $\kappa$ satisfying the product formula.

$$\prod_{\rho \in M_\kappa} \|x\|_\rho = 1, \quad x \in \kappa_*.$$

Let $S$ be a finite subset of $M_\kappa$ containing the subset of all Archimedean valuations in $M_\kappa$. We will prove the following subspace theorem on hypersurfaces:

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Theorem 1.1: *For $\rho \in S$, $i = 0, \dots, n$, let $f_{\rho i}$ be a homogeneous polynomial of degree $m \geq 1$ in $n+1$ variables with coefficients in $\kappa$ such that for each $\rho \in S$, the system*

$$f_{\rho i}(\xi) = 0, \quad i = 0, \dots, n$$

*has only the trivial solution $\xi = 0$ in $\bar{\kappa}^{n+1}$. Then for any $\varepsilon > 0$ there exist hypersurfaces $Y_1, \dots, Y_s$ of $\bar{\kappa}^{n+1}$ such that the inequality*

$$\prod_{\rho \in S} \prod_{i=0}^{n} \frac{1}{\|f_{\rho i}(\xi)\|_\rho} \leq \{\max_{\rho \in S} \|\xi\|_\rho\}^{\varepsilon} \tag{1}$$

*holds for all $S$-integral points $\xi$ in $\kappa^{n+1} - \bigcup_i Y_i$.*

In particular, if $m = 1$, the proof of the main theorem shows that the hypersurfaces $Y_1, \dots, Y_s$ in fact are hyperplanes, and so Theorem 1.1 is just Schmidt's Subspace Theorem. We even suggested Theorem 1.1 in [6] (or see [7]). Recently, P. Corvaja and U. Zannier [1] proved an analogue of Schmidt's Subspace Theorem for arbitrary polynomials in place of linear forms, in which the product of norms of $n+1$ homogeneous polynomials $f_{\rho i}(\xi)$ of $n+1$ variables in (1) is replaced by a product of $\|g_{\rho i}(x)\|_\rho^{1/\deg(g_{\rho i})}$ of $n-1$ arbitrary polynomials $g_{\rho i}$ of $n$ variables $x = (x_1, \dots, x_n)$, and the power $\varepsilon$ at the right side of (1) has the form $n - \mu + \varepsilon$, where $\mu$ is a positive number depending on degrees of the polynomials which is equal to $n-1$ if the degrees are equal. In this paper, we will utilize Corvaja–Zannier's methods to prove Theorem 1.1.

Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension $n$ defined over $\kappa$. Assume that $1 \leq n < N$. Further, let $c_{\rho i} (\rho \in S,\ i = 0, \dots, N)$ be nonnegative reals. Faltings and Wüstholz [4] proved that the set of solutions of the following system of inequalities,

$$\log\left(\frac{\|x_i\|_\rho}{\|x\|_\rho}\right) \leq -c_{\rho i} h(x),\ \rho \in S,\ i = 0, \dots, N,\ x = [x_0, \dots, x_N] \in X(\kappa), \tag{2}$$

is contained in the union of finitely many proper subvarieties of $X$ if the expectation of a particular probability distribution is larger than 1. Ferretti [5] showed that this latter condition is equivalent to

$$\frac{1}{(n+1)\deg(X)} \sum_{\rho \in S} e_X(c_\rho) > 1, \tag{3}$$

where $c_\rho = (c_{\rho 0}, \dots, c_{\rho N})$ and $e_X(c_\rho)$ is the Chow weight of $X$ with respect to $c_\rho$. If $X$ is a linear variety, then the result of Faltings and Wüstholz is equivalent to Schmidt's Subspace Theorem. Whereas Schmidt's proof of his subspace theorem is based on techniques from Diophantine approximation and geometry of numbers, Faltings and Wüstholz developed a totally different method, based on Faltings' product theorem. Using the method of Faltings and Wüstholz, Ferretti obtained a quantitative version of their result, an equivalent version of which reads as follows. Assume that

$$\frac{1}{(n+1)\deg(X)}\sum_{\rho\in S} e_X(c_\rho) > 1+\delta \tag{4}$$

with $\delta > 0$. Then there are explicitly computable constants $c_1, c_2, c_3$ depending on $N$, $n$, $\delta$, $\kappa$, $S$ and some geometry invariants of $X$ such that the set of solutions of (2) with $h(x) \geq c_1(1+h(X))$ lies in the union of at most $c_2$ proper subvarieties of $X$, each of degree $\leq c_3$. Evertse and Ferretti [3] proved another quantitative version of the result of Faltings and Wüstholz, in which the constants $c_1, c_2, c_3$ depend only $N$, $n$, $\delta$ and the degree of $X$. In particular, if the mapping $\xi \longmapsto [f_{\rho 0}(\xi), \ldots, f_{\rho n}(\xi)]$ is a finite morphism from $\mathbb{P}^n$ to $\mathbb{P}^n$ for each $\rho \in S$, then a version of Theorem 1.1 can be deduced from the result of Evertse and Ferretti.

## 2. Formalization of Theorem 1.1

To compare directly the main theorem in this paper with its analogue in Nevanlinna theory (or value distribution theory) based on the view of Vojta's dictionary [12], we first formalize Theorem 1.1 into a form (Theorem 2.2) which is just the counterpart of the second main theorem (Theorem 2.3) in Nevanlinna theory.

Let $\kappa$ be a number field and let $\rho$ be a valuation of $\kappa$. We denote by $\kappa_\rho$ the completion of $\kappa$ at $\rho$, and shall assume that the absolute values $|\cdot|_\rho$ and $\|\cdot\|_\rho$ are normalized so that

$$|p|_\rho = p^{-1}, \quad \|p\|_\rho = |p|_\rho^{[\kappa_\rho:\mathbb{Q}_p]/[\kappa:\mathbb{Q}]}$$

if $\rho|p$ and similarly for Archimedean $\rho$. Let $V_\kappa$ be a vector space of finite dimension $n+1 > 0$ over $\kappa$. Take a base $e = (e_0, \ldots, e_n)$ of $V_\kappa$. For $\xi = \xi_0 e_0 + \cdots + \xi_n e_n \in V_\kappa$, define the norm

$$|\xi|_\rho = \begin{cases} (|\xi_0|_\rho^2 + \cdots + |\xi_n|_\rho^2)^{1/2} & \text{if } \rho \text{ is Archimedean,} \\ \max_{0\leq i\leq n}\{|\xi_i|_\rho\} & \text{if } \rho \text{ is non-Archimedean.} \end{cases}$$

We will use notation

$$\|\xi\|_\rho = |\xi|_\rho^{[\kappa_\rho:\mathbb{Q}_p]/[\kappa:\mathbb{Q}]}$$

if $\rho|p$ and similarly for Archimedean $\rho$. The dual vector space $V_\kappa^*$ of $V_\kappa$ consists of all $\kappa$-linear functions $\alpha : V_\kappa \longrightarrow \kappa$, and we call $\langle \xi, \alpha \rangle = \alpha(\xi)$ the **inner product** of $\xi \in V_\kappa$ and $\alpha \in V_\kappa^*$. A norm on $V_\kappa$ induces a norm on $V_\kappa^*$.

Let $\bar{\kappa}$ be an algebraic closure of $\kappa$. Let $V = V_{\bar{\kappa}}$ be a vector space of dimension $n+1 > 0$ over $\bar{\kappa}$. Write the **projective space**

$$\mathbb{P}(V) = V/\bar{\kappa}_*$$

and let $\mathbb{P}$: $V_* \longrightarrow \mathbb{P}(V)$ be the standard **projection**, where $V_* = V - \{0\}$. If $A \subset V$, abbreviate $\mathbb{P}(A) = \mathbb{P}(A \cap V_*)$. If $\alpha \in V_*^*$, the $n$-dimensional linear subspace

$$E[a] = E[\alpha] = \mathrm{Ker}(\alpha) = \alpha^{-1}(0)$$

depends on $a = \mathbb{P}(\alpha) \in \mathbb{P}(V^*)$ only, and $\ddot{E}[a] = \mathbb{P}(E[a])$ is a **hyperplane** in $\mathbb{P}(V)$. Take $\xi \in V_*$ and set $x = \mathbb{P}(\xi)$. When the coordinates of $\xi$ and $\alpha$ are lain in $\kappa$, the distance from $x$ to $\ddot{E}[a]$ with respect to a valuation $\rho$ is defined by

$$|x, a|_\rho = \frac{|\langle \xi, \alpha \rangle|_\rho}{|\xi|_\rho |\alpha|_\rho}$$

with $0 \le |x, a|_\rho \le 1$ by using Schwarz inequality $|\langle \xi, \alpha \rangle|_\rho \le |\xi|_\rho |\alpha|_\rho$. We will use the normalization

$$\|x, a\|_\rho = |x, a|_\rho^{[\kappa_\rho:\mathbb{Q}_p]/[\kappa:\mathbb{Q}]}$$

if $\rho|p$ and similarly for Archimedean $\rho$.

Let $\bigotimes_m V$ be the $m$-fold tensor product of $V$. The set of all symmetric vectors in $\bigotimes_m V$ is a linear subspace of $\bigotimes_m V$, denoted by $\amalg_m V$, called the **$m$-fold symmetric tensor product of** $V$. For $\xi \in V_*$, let $\xi^{\amalg m}$ be the $m$-th symmetric tensor power, and define $x^{\amalg m} = \mathbb{P}(\xi^{\amalg m})$ for $x = \mathbb{P}(\xi)$. Let $V_{[m]}$ be the vector space of all homogeneous polynomials of degree $m$ on $V$. We obtain a linear isomorphism $\sim$: $\amalg_m V^* \longrightarrow V_{[m]}$ defined by

$$\tilde{\alpha}(\xi) = \langle \xi^{\amalg m}, \alpha \rangle, \quad \xi \in V,\ \alpha \in \amalg_m V^*.$$

Thus if $\xi \neq 0$ and $\alpha \neq 0$, the distance $|x^{\amalg m}, a|_\rho$ is well defined for $x^{\amalg m} = \mathbb{P}(\xi^{\amalg m})$ and $a = \mathbb{P}(\alpha)$. If $\alpha \neq 0$, the $n$-dimensional subspace $E^m[a] = \tilde{\alpha}^{-1}(0)$ in $V$ depends on $a$ only, and $\ddot{E}^m[a] = \mathbb{P}(E^m[a])$ is a **hypersurface of degree** $m$ in $\mathbb{P}(V)$. Thus $\mathbb{P}(\amalg_m V^*)$ bijectively parameterizes the hypersurfaces in $\mathbb{P}(V)$. If we identify an element $\xi$ of $V$ with its coordinates with respect to a fixed base of V, then there exist non-zero $\alpha_{\rho,i} \in \amalg_m V^*$ such that those polynomials in Theorem 1.1 can be expressed as

$$f_{\rho i}(\xi) = \tilde{\alpha}_{\rho,i}(\xi) = \langle \xi^{\amalg m}, \alpha_{\rho,i} \rangle. \tag{5}$$

Take a sequence $\{m_0, m_1, \ldots, m_q\}$ of positive integers. Let $\mathscr{A} = \{a_0, a_1, \ldots, a_q\}$ be a family of points $a_j \in \mathbb{P}(\amalg_{m_j} V^*)$. Take $\alpha_j \in \amalg_{m_j} V^* - \{0\}$ with $\mathbb{P}(\alpha_j) = a_j$, and define a homogeneous polynomial of degree $m_j$:

$$\tilde{\alpha}_j(\xi) = \langle \xi^{\amalg m_j}, \alpha_j \rangle, \quad \xi \in V, \ j = 0, 1, \ldots, q.$$

Take non-negative integers $a$ and $b$ with $a \leq b$. Let $J_a^b$ be the set of all increasing injective mappings $\lambda$: $\mathbb{Z}[0, a] \longrightarrow \mathbb{Z}[0, b]$ , where $\mathbb{Z}[0, a]$ is the set of integers $0, 1, \ldots, a$. According to Eremenko and Sodin [2], we recall the following:

*Definition 2.1:* The family $\mathscr{A} = \{a_0, a_1, \ldots, a_q\}$ $(q \geq n)$ is said to be admissible (or in general position) if, for every $\lambda \in J_n^q$, the system

$$\tilde{\alpha}_{\lambda(i)}(\xi) = 0, \quad i = 0, 1, \ldots, n \tag{6}$$

has only the trivial solution $\xi = 0$ in $V$.

Recall that $M_\kappa$ is the canonical set of distinct inequivalent valuations of $\kappa$ satisfying the product formula. We can define the **absolute height** of $\xi \in V_\kappa - \{0\}$ by

$$H(\xi) = \prod_{\rho \in M_\kappa} \|\xi\|_\rho.$$

We shall use the **absolute (logarithmic) height** $h(\xi)$ which is defined by $h(\xi) = \log H(\xi)$. Take $x \in \mathbb{P}(V)$. Then there exists $\xi \in V_*$ such that $x = \mathbb{P}(\xi)$, and so

$$H(x) = H(\xi), \quad h(x) = h(\xi)$$

are well defined. Recall the definition of $S$, which is a finite subset of $M_\kappa$ containing the subset of all Archimedean valuations in $M_\kappa$. Denote by $\mathcal{O}_{\kappa,S}$ the ring of $S$-integers of $\kappa$, i.e.,

$$\mathcal{O}_{\kappa,S} = \{z \in \kappa | \ \|z\|_\rho \leq 1, \ \rho \notin S\},$$

and denote by $\mathcal{O}_{V,S}$ the set of $S$-integral points of $V$, that is,

$$\mathcal{O}_{V,S} = \{\xi \in V | \ \|\xi\|_\rho \leq 1, \ \rho \notin S\}.$$

We will prove that Theorem 1.1 is equivalent to the following result:

Theorem 2.2: *Take $\varepsilon > 0$, $q \geq n$. Assume that for $\rho \in S$, a family*

$$\mathscr{A}_\rho = \{a_{\rho,0}, \ldots, a_{\rho,q}\} \subset \mathbb{P}(\amalg_m V^*)$$

*is admissible. Then there exist points*

$$b_i \in \mathbb{P}(\amalg_{m_i} V^*) \quad (1 \le m_i \in \mathbb{Z}, i = 1, \ldots, s < \infty)$$

*such that the inequality*

$$\sum_{\rho \in S} \sum_{j=0}^{q} \log \frac{1}{\|x^{\amalg m}, a_{\rho,j}\|_\rho} < m(n+1+\varepsilon)h(x) + O(1) \tag{7}$$

*holds for all* $x \in \mathbb{P}(V) - \bigcup_i \ddot{E}^{m_i}[b_i]$.

Originally, Theorem 2.2 was conjectured by the present authors (see [6] or [7]) based on their proof of Shiffman's conjecture (cf. [10]) over non-Archimedean fields. Based on the Corvaja–Zannier's methods, M. Ru [9] proved the Shiffman's conjecture in value distribution theory as follows:

THEOREM 2.3: *Let* $f\colon \mathbb{C}^m \longrightarrow \mathbb{P}(V_{\mathbb{C}})$ *be an algebraically non-degenerate meromorphic mapping. Fix a positive integer* $d$*. Let* $\mathscr{A} = \{a_j\}_{j=0}^q$ *be a finite admissible family of points* $a_j \in \mathbb{P}(\amalg_d V_{\mathbb{C}}^*)$ *with* $q \ge n$*. Then there exists a constant* $c > 0$ *such that Nevanlinna's characteristic function* $T_f(r)$ *and the proximity functions* $m_{f^{\amalg d}}(r, a_j)$ *of* $f$ *satisfy*

$$\sum_{j=0}^{q} m_{f^{\amalg d}}(r, a_j) \le d(n+1)T_f(r) + c \log\left\{ \left(\frac{\rho}{r}\right)^{2m-1} \frac{T_f(R)}{\rho - r} \right\} + O(1) \tag{8}$$

*for any* $r_0 < r < \rho < R$.

Based on Ye's work [13], we can improve the error term in the main inequality of Ru [9] into the form in (8) (cf. [8]). Finally, we suggest the following problem:

CONJECTURE 2.4: *Take a positive real number* $\varepsilon > 0$ *and integers* $q \ge n \ge r \ge 1$*. Assume that for* $\rho \in S$*, a family*

$$\mathscr{A}_\rho = \{a_{\rho,0}, \ldots, a_{\rho,q}\} \subset \mathbb{P}(\amalg_m V^*)$$

*is admissible. Then the set of points of* $\mathbb{P}(V) - \bigcup \ddot{E}^m[a_{\rho,j}]$ *satisfying*

$$\sum_{\rho \in S} \sum_{j=0}^{q} \log \frac{1}{\|x^{\amalg m}, a_{\rho,j}\|_\rho} \ge m(2n - r + 1 + \varepsilon)h(x) + O(1) \tag{9}$$

*is contained in a finite union of subvarieties of dimension* $\le r - 1$ *of* $\mathbb{P}(V)$.

In [6] (or see [7]), we proposed this problem for the case $r = 1$.

## 3. Equivalence of Theorem 1.1 and Theorem 2.2

We will need some basic facts. Take a positive integer $q \geq n$ and an admissible family $\mathscr{A} = \{a_0, a_1, \ldots, a_q\}$ of points $a_j \in \mathbb{P}(\amalg_{m_j} V^*)$.

$$\mathscr{A} = \{a_0, a_1, \ldots, a_q\} \ (q \geq n).$$

Let $|\cdot|$ be a norm defined over a base $e = (e_0, \ldots, e_n)$ of $V$. Write $\xi = \xi_0 e_0 + \cdots + \xi_n e_n$. By Hilbert's Nullstellensatz (cf. [11]), for each $k \in \{0, \ldots, n\}$, the identity

$$\xi_k^s = \sum_{i=0}^{n} b_{ik}^{\lambda}(\xi)\tilde{\alpha}_{\lambda(i)}(\xi) \quad (\lambda \in J_n^q) \tag{10}$$

is satisfied for some natural number $s$ with

$$s \geq m = \max_{0 \leq j \leq q} m_j,$$

where $b_{ik}^{\lambda} \in \bar{\kappa}[\xi_0, \ldots, \xi_n]$ are homogeneous polynomials of degree $s - m_{\lambda(i)}$ whose coefficients are integral-valued polynomials at the coefficients of $\tilde{\alpha}_{\lambda(i)}$ $(i = 0, \ldots, n)$. Write

$$b_{ik}^{\lambda}(\xi) = \sum_{\sigma \in J_{n, s-m_{\lambda(i)}}} b_{\sigma ik}^{\lambda} \xi_0^{\sigma(0)} \cdots \xi_n^{\sigma(n)}, \quad b_{\sigma ik}^{\lambda} \in \bar{\kappa}. \tag{11}$$

Here $J_{n,d}$ is the set of all mappings $\sigma : \mathbb{Z}[o, n] \to \mathbb{Z}[0, d]$ such that $|\sigma| = \sigma(0) + \cdots + \sigma(n) = d$.

First of all, assume that the norm $|\cdot|$ is non-Archimedean. From (10) and (11), we have

$$|\xi_k|^s \leq (\max_{i,\sigma} |b_{\sigma ik}^{\lambda}| \cdot |\alpha_{\lambda(i)}|) \max_{0 \leq i \leq n} \left\{ \frac{|\tilde{\alpha}_{\lambda(i)}(\xi)|}{|\xi|^{m_{\lambda(i)}} |\alpha_{\lambda(i)}|} \right\} |\xi|^s. \tag{12}$$

Note that

$$\max_{0 \leq k \leq n} |\xi_k|^s = |\xi|^s, \quad |\tilde{\alpha}_j(\xi)| \leq |\xi|^{m_j} |\alpha_j|. \tag{13}$$

By maximizing the inequalities (12) over $k$, $0 \leq k \leq n$, and using (13), we obtain

$$1 \leq \max_{k,i,\sigma} |b_{\sigma ik}^{\lambda}| \cdot |\alpha_{\lambda(i)}|. \tag{14}$$

Define the **gauge**

$$\Gamma(\mathscr{A}) = \min_{\lambda \in J_n^q} \min_{k,i,\sigma} \left\{ \frac{1}{|b_{\sigma ik}^{\lambda}| \cdot |\alpha_{\lambda(i)}|} \right\}, \tag{15}$$

with $0 < \Gamma(\mathscr{A}) \leq 1$. From (12), (13) and (15), we obtain

$$\Gamma(\mathscr{A}) \leq \max_{0\leq i\leq n}\left\{\frac{|\tilde{\alpha}_{\lambda(i)}(\xi)|}{|\xi|^{m_{\lambda(i)}}|\alpha_{\lambda(i)}|}\right\},$$

that is,

$$\Gamma(\mathscr{A}) \leq \max_{0\leq i\leq n}|x^{\text{II}m_{\lambda(i)}}, a_{\lambda(i)}|, \quad \lambda \in J_n^q, \quad x \in \mathbb{P}(V). \tag{16}$$

If the norm $|\cdot|$ is Archimedean, now (10) and (11) imply

$$|\xi_k|^s \leq \left(\sum_{i=0}^{n}\sum_{\sigma}|b_{\sigma ik}^{\lambda}|\cdot|\alpha_{\lambda(i)}|\right)\max_{0\leq i\leq n}\left\{\frac{|\tilde{\alpha}_{\lambda(i)}(\xi)|}{|\xi|_*^{m_{\lambda(i)}}|\alpha_{\lambda(i)}|}\right\}|\xi|_*^s, \tag{17}$$

where $|\xi|_* = \max_k |\xi_k|$. W.l.o.g., we may assume

$$|\xi| = (|\xi_0|^2 + \cdots + |\xi_n|^2)^{1/2}.$$

Since $|\xi| \leq \sqrt{n+1}|\xi|_*$, then (17) yields

$$1 \leq (n+1)^{m/2}\max_k\sum_{i=0}^{n}\sum_{\sigma}|b_{\sigma ik}^{\lambda}|\cdot|\alpha_{\lambda(i)}|. \tag{18}$$

Define the **gauge**

$$\Gamma(\mathscr{A}) = (n+1)^{-m/2}\min_{\lambda\in J_n^q}\min_k\left\{\sum_{i=0}^{n}\sum_{\sigma}|b_{\sigma ik}^{\lambda}|\cdot|\alpha_{\lambda(i)}|\right\}^{-1}, \tag{19}$$

with $0 < \Gamma(\mathscr{A}) \leq 1$. From (17) and (19), we obtain the inequality (16).

Lemma 3.1: *For $x \in \mathbb{P}(V)$, $r \in \mathbb{R}$, define*

$$\mathscr{A}(x, r) = \{j \mid |x^{\text{II}m_j}, a_j| < r,\ 0 \leq j \leq q\}. \tag{20}$$

*If $0 < r \leq \Gamma(\mathscr{A})$, then $\#\mathscr{A}(x, r) \leq n$.*

*Proof:* Assume that $\#\mathscr{A}(x, r) \geq n + 1$. Then $\lambda \in J_n^q$ exists such that

$$\{\lambda(0), \ldots, \lambda(n)\} \subseteq \mathscr{A}(x, r).$$

Hence

$$|x^{\text{II}m_{\lambda(i)}}, a_{\lambda(i)}| < r \leq \Gamma(\mathscr{A}), \quad i = 0, \ldots, n,$$

which is impossible according to (16). ∎

LEMMA 3.2: *Take $x \in \mathbb{P}(V)$ such that $|x^{\amalg m_j}, a_j| > 0$ for $j = 0, \ldots, q$. Then*

$$\prod_{j=0}^{q} \frac{1}{|x^{\amalg m_j}, a_j|} \leq \left(\frac{1}{\Gamma(\mathscr{A})}\right)^{q-n} \max_{\lambda \in J_n^q} \left\{ \prod_{i=0}^{n} \frac{1}{|x^{\amalg m_{\lambda(i)}}, a_{\lambda(i)}|} \right\} \tag{21}$$

$$\leq \left(\frac{1}{\Gamma(\mathscr{A})}\right)^{q+1-n} \max_{\lambda \in J_{n-1}^q} \left\{ \prod_{i=0}^{n-1} \frac{1}{|x^{\amalg m_{\lambda(i)}}, a_{\lambda(i)}|} \right\}. \tag{22}$$

*Proof:* Take $r = \Gamma(\mathscr{A})$. Lemma 3.1 implies $\#\mathscr{A}(x, r) \leq n$. Thus $\sigma \in J_n^q$ exists such that $\mathscr{A}(x, r) \subseteq \{\sigma(0), \ldots, \sigma(n)\}$. Note that $\mathrm{Im}\lambda - \mathscr{A}(x, r) \neq \emptyset$ for any $\lambda \in J_n^q$. Then we have

$$\begin{aligned} \prod_{j=0}^{q} \frac{1}{|x^{\amalg m_j}, a_j|} &\leq r^{n-q} \prod_{i=0}^{n} \frac{1}{|x^{\amalg m_{\sigma(i)}}, a_{\sigma(i)}|} \\ &\leq \left(\frac{1}{\Gamma(\mathscr{A})}\right)^{q-n} \max_{\lambda \in J_n^q} \left\{ \prod_{i=0}^{n} \frac{1}{|x^{\amalg m_{\lambda(i)}}, a_{\lambda(i)}|} \right\} \\ &\leq \left(\frac{1}{\Gamma(\mathscr{A})}\right)^{q+1-n} \max_{\lambda \in J_{n-1}^q} \left\{ \prod_{i=0}^{n-1} \frac{1}{|x^{\amalg m_{\lambda(i)}}, a_{\lambda(i)}|} \right\}. \qquad \blacksquare \end{aligned}$$

LEMMA 3.3 (cf. [6]): *For $x \in \mathbb{P}(V)$, we can choose $\xi \in \mathcal{O}_{V,S}$ such that $x = \mathbb{P}(\xi)$, and the absolute height of $x$ satisfies*

$$\max\left\{ \max_{\rho \in S} \|\xi\|_\rho, \prod_{\rho \in S} \|\xi\|_\rho \right\} \leq cH(x) \leq c\{\max_{\rho \in S} \|\xi\|_\rho\}^{\#S},$$

*where $c$ is a constant depending only on $S$ but not on $x$.*

Obviously, Theorem 2.2 yields immediately Theorem 1.1 by taking $q = n$ and using Lemma 3.3. Conversely, Theorem 1.1 implies Theorem 2.2. In fact, by Lemma 3.2 and Theorem 1.1, there exist points

$$b_i \in \mathbb{P}(\amalg_{m_i} V^*) \quad (1 \leq m_i \in \mathbb{Z},\ i = 1, \ldots, s < \infty)$$

such that the inequality

$$\begin{aligned} \prod_{\rho \in S} \prod_{j=0}^{q} \frac{1}{\|x^{\amalg m}, a_{\rho,j}\|_\rho} &\leq \prod_{\rho \in S} \left\{ \left(\frac{1}{\Gamma(\mathscr{A}_\rho)}\right)^{q-n} \prod_{j=0}^{n} \frac{1}{\|x^{\amalg m}, a_{\rho,\sigma_\rho(j)}\|_\rho} \right\} \\ &\leq c_1 \left( \prod_{\rho \in S} \|\xi\|_\rho^m \right)^{n+1} \left( \prod_{\rho \in S} \prod_{j=0}^{n} \frac{1}{\|\langle \xi^{\amalg m}, \alpha_{\rho,\sigma_\rho(j)} \rangle\|_\rho} \right) \\ &\leq c_1 \left( \prod_{\rho \in S} \|\xi\|_\rho \right)^{m(n+1)} (\max_{\rho \in S} \|\xi\|_\rho)^\varepsilon \end{aligned}$$

holds for all points $x = \mathbb{P}(\xi) \in \mathbb{P}(V) - \bigcup_i \ddot{E}^{m_i}[b_i]$, where $c_1$ is constant, and $\alpha_{\rho,j} \in \amalg_m V^* - \{0\}$ with $a_{\rho,j} = \mathbb{P}(\alpha_{\rho,j})$. By Lemma 3.3, there exists a constant $c_2$ such that

$$\prod_{\rho\in S}\prod_{j=0}^{q} \frac{1}{|||x^{\amalg m}, a_{\rho,j}|||_\rho} \leq c_2 H(x)^{m(n+1)+\varepsilon},$$

and hence Theorem 2.2 follows.

## 4. Proof of Theorem 2.2

We now proceed with the proof of Theorem 2.2 which will be based on the methods of P. Corvaja and U. Zannier [1]. First of all, we recall several lemmas from [1]. We shall use the **lexicographic ordering** on the $p$-tuples $\nu = (\nu(1), \ldots, \nu(p)) \in \mathbb{Z}_+^p$, namely, $\mu > \nu$ if and only if for some $l \in \{1, \ldots, p\}$ we have $\mu(k) = \nu(k)$ for $k < l$ and $\mu(l) > \nu(l)$.

Lemma 4.1: *Let $A$ be a commutative ring and let $\{g_1, \ldots, g_p\}$ be a regular sequence in $A$. Suppose that for some $y, x_1, \ldots, x_h \in A$ we have an equation*

$$g_1^{\nu(1)} \cdots g_p^{\nu(p)} y = \sum_{k=1}^{h} g_1^{\mu_k(1)} \cdots g_p^{\mu_k(p)} x_k,$$

*where $\mu_k > \nu$ for $k = 1, \ldots, h$. Then $y \in I_p = (g_1, \ldots, g_p)$, the ideal generated by $g_1, \ldots, g_p$.*

Lemma 4.2: *Let $\tilde{\beta}_1, \ldots, \tilde{\beta}_p$ be homogeneous polynomials in $\bar{\kappa}[\xi_0, \ldots, \xi_n]$. Assume that they define a subvariety of $\mathbb{P}(V)$ of dimension $n-p$. Then $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_p\}$ is a regular sequence.*

Lemma 4.3: *Let $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ be homogeneous polynomials in $\bar{\kappa}[\xi_0, \ldots, \xi_n]$. Assume that they define a subvariety of $\mathbb{P}(V)$ of dimension 0. Then, for all large $N$,*

$$\dim V_{[N]}/\{(\tilde{\beta}_1, \ldots, \tilde{\beta}_n) \cap V_{[N]}\} = \deg(\tilde{\beta}_1) \cdots \deg(\tilde{\beta}_n).$$

Take $\rho \in S$ and take a positive integer $d$. Let $\mathscr{A}_\rho = \{a_{\rho,j}\}_{j=0}^q$ be a finite admissible family of points $a_{\rho,j} \in \mathbb{P}(\amalg_d V^*)$ with $q \geq n$. Take $\alpha_{\rho,j} \in \amalg_d V^* - \{0\}$ with $\mathbb{P}(\alpha_{\rho,j}) = a_{\rho,j}$, and define

$$\tilde{\alpha}_{\rho,j}(\xi) = \langle \xi^{\amalg d}, \alpha_{\rho,j}\rangle, \quad \xi \in V,\ j = 0, 1, \ldots, q.$$

W.l.o.g., assume $|\alpha_{\rho,j}| = 1$ for $j = 0, \ldots, q$. Lemma 3.2 implies

$$\prod_{j=0}^{q} \frac{1}{\|x^{\amalg d}, a_{\rho,j}\|_\rho} \leq \left(\frac{1}{\Gamma(\mathscr{A}_\rho)}\right)^{q+1-n} \max_{\lambda\in J_{n-1}^q} \prod_{i=0}^{n-1} \frac{1}{\|x^{\amalg d}, a_{\rho,\lambda(i)}\|_\rho} \tag{23}$$

for $x \in \mathbb{P}(V) - \bigcup_{j=0}^{q} \ddot{E}^d[a_{\rho,j}]$. According to P. Corvaja and U. Zannier [1], we will estimate the term in the right sides of (23) as follows.

Now pick $\lambda \in J_{n-1}^q$. Since $\mathscr{A}_\rho$ is admissible, then $\tilde{\alpha}_{\rho,\lambda(0)}, \ldots, \tilde{\alpha}_{\rho,\lambda(n-1)}$ define a subvariety of $\mathbb{P}(V)$ of dimension 0. Take a multi-index $\nu = (\nu(1), \ldots, \nu(n)) \in \mathbb{Z}_+^n$ with length

$$|\nu| = \nu(1) + \cdots + \nu(n) \leq N/d.$$

For any $\gamma = (\gamma(1), \ldots, \gamma(n)) \in \mathbb{Z}_+^n$, abbreviate

$$\tilde{\alpha}_{\rho,\lambda}^{\gamma} = \tilde{\alpha}_{\rho,\lambda(0)}^{\gamma(1)} \cdots \tilde{\alpha}_{\rho,\lambda(n-1)}^{\gamma(n)}$$

and define the spaces

$$\mathbf{V}_{N,\nu} = \sum_{\gamma \geq \nu} \tilde{\alpha}_{\rho,\lambda}^{\gamma} V_{[N-d|\gamma|]}$$

with $\mathbf{V}_{N,0} = V_{[N]}$ and $\mathbf{V}_{N,\mu} \subset \mathbf{V}_{N,\nu}$ if $\mu > \nu$. Thus the $\mathbf{V}_{N,\nu}$ define a filtration of $V_{[N]}$. Next we consider quotients between consecutive spaces in the filtration.

Lemma 4.4: *Suppose that $\mathbf{V}_{N,\mu}$ follows $\mathbf{V}_{N,\nu}$ in the filtration:*

$$V_{[N]} \supset \cdots \supset \mathbf{V}_{N,\nu} \supset \mathbf{V}_{N,\mu} \supset \cdots \supset \{0\}. \tag{24}$$

*Then there is an isomorphism*

$$\mathbf{V}_{N,\nu}/\mathbf{V}_{N,\mu} \cong V_{[N-d|\nu|]}/\{(\tilde{\alpha}_{\rho,\lambda(0)}, \ldots, \tilde{\alpha}_{\rho,\lambda(n-1)}) \cap V_{[N-d|\nu|]}\}.$$

By Lemma 4.3 and Lemma 4.4, there exists an integer $N_0$ depending only on $\tilde{\alpha}_{\rho,\lambda(0)}, \ldots, \tilde{\alpha}_{\rho,\lambda(n-1)}$ such that

$$\Delta_\nu := \dim \mathbf{V}_{N,\nu}/\mathbf{V}_{N,\mu} \begin{cases} = d^n, & \text{if } d|\nu| < N - N_0; \\ \leq \dim V_{[N_0]}, & \text{otherwise.} \end{cases} \tag{25}$$

Now we choose inductively a suitable basis of $V_{[N]}$ in the following way. We start with the last nonzero $\mathbf{V}_{N,\mu}$ in the filtration (24) and pick any basis of it. Suppose $\mu > \nu$ are consecutive $n$-tuples such that $d|\nu|, d|\mu| \leq N$. It follows directly from the definition that we may pick representatives $\tilde{\alpha}_{\rho,\lambda}^{\nu}\tilde{\beta} \in \mathbf{V}_{N,\nu}$ of elements from the quotient space $\mathbf{V}_{N,\nu}/\mathbf{V}_{N,\mu}$, where $\tilde{\beta} \in V_{[N-d|\nu|]}$. We extend the previously constructed basis in $\mathbf{V}_{N,\mu}$ by adding these representatives. In particular, we have obtained a basis for $\mathbf{V}_{N,\nu}$ and our inductive procedure may go on unless $\mathbf{V}_{N,\nu} = V_{[N]}$, in which case we stop. In this way, we obtain a basis $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_M\}$ of $V_{[N]}$, where $M = \dim V_{[N]}$.

For a fixed $k \in \{1, \ldots, M\}$, assume that $\tilde{\psi}_k$ is constructed with respect to $\mathbf{V}_{N,\nu}/\mathbf{V}_{N,\mu}$. We may write $\tilde{\psi}_k = \tilde{\alpha}_{\rho,\lambda}^{\nu}\tilde{\beta}$ with $\tilde{\beta} \in V_{[N-d|\nu|]}$. Then we have a bound

$$\|\tilde{\psi}_k(\xi)\|_\rho = \|\tilde{\alpha}_{\rho,\lambda}^{\nu}(\xi)\|_\rho \|\tilde{\beta}(\xi)\|_\rho \leq c' \|\tilde{\alpha}_{\rho,\lambda}^{\nu}(\xi)\|_\rho \|\xi\|_\rho^{N-d|\nu|},$$

where $c'$ is a positive constant depending only on $\tilde{\psi}_k$, not on $\xi$. Observe that there are precisely $\Delta_\nu$ such functions $\tilde{\psi}_k$ in our basis. Hence, taking the product over all functions in the basis, and then taking logarithms, we get

$$\begin{aligned}\log \prod_{k=1}^{M} \|\tilde{\psi}_k(\xi)\|_\rho \leq & \sum_{\nu} \sum_{i=0}^{n-1} \Delta_\nu \nu(i+1) \log \|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho \\ & + \left( \sum_{\nu} \Delta_\nu (N - d|\nu|) \right) \log \|\xi\|_\rho + c,\end{aligned} \tag{26}$$

where $c$ is a positive constant depending only on $\tilde{\psi}_k$, not on $\xi$. Here $\nu$ in the summation ranges over the $n$-tuples in the filtration (24) with $|\nu| \leq N/d$.

Note that

$$M = \dim V_{[N]} = \binom{n+N}{N} = \frac{N^n}{n!} + O(N^{n-1}), \tag{27}$$
$$\sum_{t=0}^{T} \# J_{n-1,t} = \# J_{n,T} = \binom{n+T}{T}, \quad T \in \mathbb{Z}^+,$$

and that, since the sum below is independent of $j$, we have that, for any positive integer $T$ and for every $0 \leq j \leq n$,

$$\begin{aligned}\sum_{\nu \in J_{n,T}} \nu(j) &= \frac{1}{n+1} \sum_{\nu \in J_{n,T}} \sum_{j=0}^{n} \nu(j) = \frac{1}{n+1} \sum_{\lambda \in J_{n,T}} T \\ &= \frac{T}{n+1} \binom{n+T}{T} = \frac{T^{n+1}}{(n+1)!} + O(T^n).\end{aligned} \tag{28}$$

Then, for $N$ divisible by $d$ and for every $0 \leq i \leq n-1$, (25) and (28) with $T = N/d$ yield

$$\sum_{\nu} \Delta_\nu \nu(i+1) = \frac{N^{n+1}}{d(n+1)!} + O(N^n), \tag{29}$$

which implies

$$\sum_{\nu} \Delta_\nu d|\nu| = d \sum_{i=0}^{n-1} \sum_{\nu} \Delta_\nu \nu(i+1) = \frac{nN^{n+1}}{(n+1)!} + O(N^n). \tag{30}$$

Note that

$$\sum_{\nu} \Delta_\nu N = \frac{N^{n+1}}{n!} + O(N^n).$$

Therefore, by (26), (27), (29) and (30), we have

$$\begin{aligned}\log\prod_{k=1}^{M}\|\tilde{\psi}_k(\xi)\|_\rho \leq& \frac{N^{n+1}}{d(n+1)!}\Big(1+O\Big(\frac{1}{N}\Big)\Big)\log\prod_{i=0}^{n-1}\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho \\ &+\frac{N^{n+1}}{(n+1)!}\Big(1+O\Big(\frac{1}{N}\Big)\Big)\log\|\xi\|_\rho + c \\ \leq& K\Big\{\log\prod_{i=0}^{n-1}\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho + d\log\|\xi\|_\rho\Big\}+c,\end{aligned} \tag{31}$$

where $K=K(d,n,N)$ is a positive constant such that

$$K=\frac{N^{n+1}}{d(n+1)!}\Big(1+O\Big(\frac{1}{N}\Big)\Big). \tag{32}$$

Let $\tilde{\phi}_1,\ldots,\tilde{\phi}_M$ be a fixed basis of $V_{[N]}$ such that, when $\xi\in V-\{0\}$,

$$\Xi=(\tilde{\phi}_1(\xi),\ldots,\tilde{\phi}_M(\xi))\in\mathbb{C}^M-\{0\}.$$

Then $\tilde{\psi}_k$ can be expressed as a linear form $L_k$ in $\tilde{\phi}_1,\ldots,\tilde{\phi}_M$ so that $\tilde{\psi}_k(\xi)=L_k(\Xi)$. The linear forms $L_1,\ldots,L_M$ are linearly independent. By (31), we obtain

$$\begin{aligned}\log\prod_{k=1}^{M}\|L_k(\Xi)\|_\rho &\leq K\Big\{\log\prod_{i=0}^{n-1}\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho + d\log\|\xi\|_\rho\Big\}+c \\ &= K\Big\{\log\prod_{i=0}^{n-1}\frac{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho}{\|\xi\|_\rho^d}+(n+1)d\log\|\xi\|_\rho\Big\}+c,\end{aligned}$$

which implies

$$\log\prod_{i=0}^{n-1}\frac{\|\xi\|_\rho^d}{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho}\leq\frac{1}{K}\Big\{\log\prod_{k=1}^{M}\frac{1}{\|L_k(\Xi)\|_\rho}+c\Big\}+(n+1)d\log\|\xi\|_\rho, \tag{33}$$

or, equivalently,

$$\prod_{i=0}^{n-1}\frac{\|\xi\|_\rho^d}{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho}\leq\Big\{e^c\prod_{k=1}^{M}\frac{1}{\|L_k(\Xi)\|_\rho}\Big\}^{1/K}\|\xi\|_\rho^{(n+1)d}. \tag{34}$$

Fix $\varepsilon>0$. By Schmidt's Subspace Theorem for linear forms (see [12]), for all $\lambda\in J_{n-1}^q$, the set $Q$ of all $\Xi\in\mathcal{O}_{V_{[N]},S}$ satisfying

$$\prod_{\rho\in S}\prod_{k=1}^{N}\|L_k(\Xi)\|_\rho<\{\max_{\rho\in S}\|\Xi\|_\rho\}^{-\varepsilon}$$

is contained in a finite union of hyperplanes of $V_{[N]}$. Note that $Q$ is just a finite union of hypersurfaces of degree $N$ in $V$, say,

$$Q = \bigcup_{j=1}^{r} E^N[b_j], \quad b_j \in \mathbb{P}(\amalg_N V^*),$$

and that there is a positive constant $\tilde{c}$ such that $\|\Xi\|_\rho \le \tilde{c}\|\xi\|_\rho^N$ for $\rho \in S$. Then

$$\prod_{\rho\in S}\prod_{i=0}^{n-1} \frac{\|\xi\|_\rho^d}{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho} \le \{e^c(\max_{\rho\in S}\tilde{c}\|\xi\|_\rho^N)^\varepsilon\}^{1/K}\left(\prod_{\rho\in S}\|\xi\|_\rho\right)^{(n+1)d},$$

where

$$\xi \notin \bigcup_{\rho\in S}\bigcup_{j=0}^{q} E^d[a_{\rho,j}] \cup Q.$$

If we choose $N$ large enough such that $N/K \le 1$, then Lemma 3.3 implies that there is a constant $c$ depending only on $S$ but not on $\xi$ such that

$$\prod_{\rho\in S}\prod_{i=0}^{n-1} \frac{\|\xi\|_\rho^d}{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho} \le cH(\xi)^{(n+1+\varepsilon)d}. \tag{35}$$

Therefore, Theorem 2.2 follows from (23) and (35).

*Remark on (35):* If we take $\lambda \in J_n^q$, Lemma 3.1 means that there exists an index $i_0 \in \{0, 1, \ldots, n\}$ such that

$$\|x^{\amalg d}, a_{\rho,\lambda(i_0)}\|_\rho \ge \Gamma(\mathscr{A}_\rho), \quad x = \mathbb{P}(\xi).$$

W.l.o.g., we may assume $i_0 = n$. Thus from (34), according to the arguments of (35) we can obtain

$$\prod_{\rho\in S}\prod_{i=0}^{n} \frac{1}{\|\tilde{\alpha}_{\rho,\lambda(i)}(\xi)\|_\rho} \le c(\max_{\rho\in S}\|\xi\|_\rho)^\varepsilon. \tag{36}$$

Thus the above methods will yield a proof of Theorem 1.1 as well.

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